APPENDIX

Mathematical Modelling

A.I.1 Introduction

On 25th February 2013, the ISRO launcher PSLV C20, put the satellite SARAL into orbit. The satellite weighs 407 kg. It is at an altitude of 781 km and its orbit is inclined at an angle of 98.5º.

On reading the above information, we may wonder:

(i) How did the scientists calculate the altitude as 781km. Did they go to space and measure it?

(ii) How did they conclude that the angle of orbit is 98.5º without actually measuring?

Some more examples are there in our daily life where we wonder how the scientists and mathematicians could possibly have estimated these results. Observe these examples:

(i) The temperature at the surface of the sun is about 6,000ºC.

(ii) The human heart pumps 5 to 6 liters of blood in the body every minute.

(iii) We know that the distance between the sun and the earth is 1,49,000 km.

In the above examples, we know that no one went to the sun to measure the temperature or the distance from earth. Nor can we take the heart out of the body and measure the blood it pumps. The way we answer these and other similar questions is through mathematical modelling.

Mathematical modelling is used not only by scientists but also by us. For example, we might want to know how much money we will get after one year if we invest ₹100 at 10% simple interest. Or we might want to know how many litres of paint is needed to whitewash a room. Even these problems are solved by mathematical modelling.

Think - Discuss

Discuss with your friends some more examples in real life where we cannot directly measure and must use mathematical modelling .

A.I.2 Mathematical Models

Do you remember the formula to calculate the area of a triangle?

Area of Triangle = $\frac{1}{2} \times$ base $\times$ height.

Similarly, simple interest calculation uses the formula $I = \frac{PTR}{100}$. This formula or equation is a relation between the Interest (I); Principle (P); Time (T); and Rate of Interest (R). These formulae are examples of mathematical models.

Some more examples for mathematical models.

(i) Speed (S) $= \frac{Distance\ (d)}{time\ (t)}$

(ii) In compound interest sum (A) = $P\left(1+\frac{r}{100}\right)^n$

Where P = Principle

r = rate of interest

n = no. of times to be calculated interest.

So, Mathematical model is nothing but a mathematical description or relation that describes some real life situation.

Do This

Write some more mathematical models which you have learnt in previous classes.

A.I.3 Mathematical Modelling

We often face problems in our day to day life. To solve them, we try to write it as an equivalent mathematical problem and find its solution. Next we interpret the solution and check to what extent the solution is valid. This process of constructing a mathematical model and using it to find the answer is known as mathematical modelling.

Now we have to observe some more examples related to mathematical modelling.

Example-1. Vani wants to buy a TV that costs ₹19,000 but she has only ₹15,000. So she decides to invest her money at 8% simple interest per year. After how many years will she be able to buy the TV?

Step 1 : (Understanding the problem): In this stage, we define the real problem. Here, we are given the principal, the rate of simple interest and we want to find out the number of years after which the amount will become Rs. 19000.

Step 2 : (Mathematical description and formulation) In this step, we describe, in mathematical terms, the different aspects of the problem. We define variables, write equations or inequalities and gather data if required.

Here, we use the formula for simple interest which is

$$I = \frac{PTR}{100} \text{ (Model)}$$

where P = Principle, T = number of years, R = rate of interest, I = Interest

We need to find time $= T = \frac{100I}{RP}$

Step 3: (Solving the mathematical problem) In this step, we solve the problem using the formula which we have developed in step 2.

We know that Vani already has ₹15,000 which is the principal, P

The final amount is ₹19000 so she needs an extra (19000-15000) = ₹4000. This will come from the interest, I.

P = ₹15,000, Rate = 8%, then I = 4000; $\text{T} = \frac{100 \times 4\not{0}\not{0}\not{0}}{150\not{0}\not{0} \times 8} = \frac{40\not{0}}{12\not{0}}$

$$\text{T} = 3\frac{4}{12} = 3\frac{1}{3} \text{ years}$$

or **Step4 : (Interpreting the solution):** The solution obtained in the previous step is interpreted here.

Here T = $3\frac{1}{3}$. This means three and one third of a year or three years and 4 months.

So, Vani can buy a washing machine after 3 year 4 months

Step5 : (Validating the model): We can't always accept a model that gives us an answer that does not match the reality. The process of checking and modifying the mathematical model, if necessary, is validation.

In the given example, we are assuming that the rate of interest will not change. If the rate changes then our model $\frac{PTR}{100}$ will not work. We are also assuming that the price of the washing machine will remain Rs. 19,000.

Let us take another example.

Example-2. In Lokeshwaram High school, 50 children in the 10th class and their Maths teacher want to go on tour from Lokeshwaram to Hyderabad by vehicles. Each vehicle can hold six persons not including driver. How many vehicles they need to hire?

Step 1 : We want to find the number of vehicles needed to carry 51 persons, given that each jeep can seat 6 persons besides the driver.

Step 2 : Number of vehicles = (Number of persons) / (Persons that can be seated in one jeep)

Step 3 : Number of vehicles = 51/6 = 8.5

Step 4 : Interpretation

We know that it is not possible to have 8.5 vehicles. So, the number of vehicles needed has to be the nearest whole number which is 9.

$\therefore$ Number of vehicles need is 9.

Step 5 : Validation

While modelling, we have assumed that lean and fat children occupy same space.

Do This

1. Take any word problem from your textbook, make a mathematical model for the chosen problem and solve it.
2. Make a mathematical model for the problem given below and solve it.

   Suppose a car starts from a place A and travels at a speed of 40 Km/h towards another place B. At the same time another car starts from B and travels towards A at a speed of 30 Km/h. If the distance between A and B is 100 km; after how much time will that cars meet?

So far, we have made mathematical models for simple word problems. Let us take a real life example and model it.

Example-3. In the year 2000, 191 member countries of the U.N.signed a declaration to promote gender equality. One indicator for deciding whether this goal has been achieved is the ratio of girls to boys in primary, secondary education. India also signed the declaration. The data for the percentage of girls in India who are enrolled in primary schools is given in Table **A.I.1.**

Table A.I.1

Year	Enrolment (in %)
1991 – 92	41.9
1992 – 93	42.6
1993 – 94	42.7
1994 – 95	42.9
1995 – 96	43.1
1996 – 97	43.2
1997 -98	43.5
1998 – 99	43.5
1999 – 2000	43.6
2000 – 01	43.7
2001 - 02	44.1

Using this data, mathematically describe the rate at which the proportion of girls enrolled in primary schools grew. Also, estimate the year by which the enrollment of girls will reach 50%.

Solution :

Step 1 : Formulation Let us first convert the problem into a mathematical problem.

Table A.I.1 gives the enrollment for the years 1991 – 92, 1992- 93 etc. Since the students join at the begining of an academic year, we can take the years as 1991, 1992 etc. Let us assume that the percentage of girls who join primary schools will continue to grow at the same rate as the rate in Table A.I.1. So, the number of years is important, not the specific years. (To give a similar situation, when we find the simple interest for say, ₹ 15000 at the rate 8% for three years, it does not matter whether the three – year period is from 1999 to 2002 or from 2001 to 2004. What is important is the interest rate in the years being considered)

Here also, we will see how the enrollment grows after 1991 by comparing the number of years that has passed after 1991 and the enrollment. Let us take 1991 as the 0^{th} year, and write 1 for 1992 since 1 year has passed in 1992 after 1991. Similarly we will write 2 for 1993, 3 for 1994 etc. So, Table A.I.1 will now look like as Table A.I.2

Table A.I.2

Year	Enrolment (in%)
0	41.9
1	42.6
2	42.7
3	42.9
4	43.1
5	43.2
6	43.5
7	43.5
8	43.6
9	43.7
10	44.1

The increase in enrolment is given in the following table A.I.3.

Table A.I.3

Year	Enrolment (in%)	Increase
0	41.9	0
1	42.6	0.7
2	42.7	0.1
3	42.9	0.2
4	43.1	0.2
5	43.2	0.1
6	43.5	0.3
7	43.5	0
8	43.6	0.1
9	43.7	0.1
10	44.1	0.4

At the end of the first year period from 1991 to 1992, the enrollment has increased by 0.7% from 41.9% to 42.6%. At the end of the second year, this has increased by 0.1% from 42.6% to 42.7%. From the table above, we cannot find a definite relationship between the number of years and percentage. But the increase is fairly steady. Only in the first year and in the 10^{th} year there is a jump. The mean of these values is

$$\frac{0.7+0.1+0.2+0.2+0.1+0.3+0+0.1+0.1+0.4}{10} = 0.22 \qquad \text{.... (1)}$$

Let us assume that the enrolment steadily increases at the rate of 0.22 percent.

Step 2 : (Mathematical Description)

We have assumed that the enrolment increases steadily at the rate of 0.22% per year.

So, the Enrolment Percentage (EP) in the first year = 41.9 + 0.22

EP in the second year = 41.9 + 0.22 + 0.22 = 41.9 + 2 $\times$ 0.22

EP in the third year = 41.9 + 0.22 + 0.22 + 0.22 = 41.9 + 3 $\times$ 0.22

So, the enrolment percentage in the n^{th} year = 41.9 + 0.22n, for n $\geq$ 1. .... (2)

Now, we also have to find the number of years by which the enrolment will reach 50%. So, we have to find the value of *n* from this equation

$$50 = 41.9 + 0.22n$$

Step 3 : Solution : Solving (2) for n, we get

$$n = \frac{50-41.9}{0.22} = \frac{8.1}{0.22} = 36.8$$

Step 4 : (Interpretation) : Since the number of years is an integral value, we will take the next higher integer, 37. So, the enrolment percentage will reach 50% in 1991 + 37 = 2028.

Step 5 : (Validation) Since we are dealing with a real life situation, we have to see to what extent this value matches the real situation.

Let us check Formula (2) is in agreement with the reality. Let us find the values for the years we already know, using Formula (2), and compare it with the known values by finding the difference. The values are given in Table A.I.4.

Table A.I.4

Year	Enrolment (in %)	Values given by (2) (in %)	Difference (in %)
0	41.9	41.90	0
1	42.6	42.12	0.48
2	42.7	42.34	0.36
3	42.9	42.56	0.34
4	43.1	42.78	0.32
5	43.2	43.00	0.20
6	43.5	43.22	0.28
7	43.5	43.44	0.06
8	43.6	43.66	-0.06
9	43.7	43.88	-0.18
10	44.1	44.10	0.00

As you can see, some of the values given by Formula (2) are less than the actual values by about 0.3% or even by 0.5%. This can give rise to a difference of about 3 to 5 years since the increase per year is actually 1% to 2%. We may decide that this much of a difference is acceptable and stop here. In this case, (2) is our mathematical model.

Suppose we decide that this error is quite large, and we have to improve this model. Then, we have to go back to Step 2, and change the equation. Let us do so.

Step 1 : Reformulation : We still assume that the values increase steadily by 0.22%, but we will now introduce a correction factor to reduce the error, For this, we find the mean of all the errors. This is

$$\frac{0+0.48+0.36+0.34+0.32+0.2+0.28+0.06-0.06-0.18+0}{10} = 0.18$$

We take the mean of the errors, and correct our formula by this value.

Revised Mathematical Description : Let us now add the mean of the errors to our formula for enrolment percentage given in (2). So, our corrected formula is :

Enrolment percentage in the nth year

$$= 41.9 + 0.22n + 0.18 = 42.08 + 0.22n, \text{ for } n \geq 1 \qquad \ldots (3)$$

We will also modify Equation (2) appropriately. The new equation for n is :

$$50 = 42.08 + 0.22n \qquad \ldots (4)$$

Altered Solution : Solving Equation (4) for n, we get

$$n = \frac{50 - 42.08}{0.22} = \frac{7.92}{0.22} = 36$$

Interpretation : Since n = 36, the enrolment of girls in primary schools will reach 50% in the year 1991 + 36 = 2027.

Validation : Once again, let us compare the values got by using Formula (4) with the actual values. Table A.I.5 gives the comparison.

Table A.I.5

Year	Enrolment (in %)	Values given by (2)	Difference between Values	Values given by (4)	Difference between values
0	41.9	41.90	0	41.9	0
1	42.6	42.12	0.48	42.3	0.3
2	42.7	42.34	0.36	42.52	0.18
3	42.9	42.56	0.34	42.74	0.16
4	43.1	42.78	0.32	42.96	0.14
5	43.2	43.00	0.20	43.18	0.02
6	43.5	43.22	0.28	43.4	0.1
7	43.5	43.44	0.06	43.62	-0.12
8	43.6	43.66	-0.06	43.84	-0.24
9	43.7	43.88	-0.18	44.06	-0.36
10	44.1	44.10	0.00	44.28	-0.18

As you can see, many of the values that (4) gives are closer to the actual value than the values that (2) gives. The mean of the errors is 0 in this case.

A.I.4 Advantages of Mathematics Modeling

1. The aim of mathematical modeling is to get some useful information about a real world problem by converting it into mathematical problem. This is especially useful when it is not possible or very expensive to get information by other means such as direct observation or by conducting experiments.

   For example, suppose we want to study the corrosive effect of the discharge of the Mathura refinery on the Taj Mahal. We would not like to carry out experiments on the Taj Mahal directly because that would damage a valuable monument. Here mathematical modeling can be of great use.

2. Forecasting is very important in many types of organizations, since predictions of future events have to be incorporated into the decision – making process.

For example

(i) In marketing departments, reliable forecasts of demand help in planning of the sale strategies

(ii) A school board needs to able to forecast the increase in the number of school going children in various districts so as to decide where and when to start new schools.

3. Often we need to estimate large values like trees in a forest; fishes in a lake; estimation of votes polled etc.

   Some more examples where we use mathematical modelling are:

   (i) Estimating future population for certain number of years

   (ii) Predicting the arrival of Monsoon

   (iii) Estimating the literacy rate in coming years

   (iv) Estimating number of leaves in a tree

   (v) Finding the depth of oceans

A.I.5 Limitations of Mathematical modeling

Is mathematical modeling the answer to all our problems?

Certainly not; it has it's limitations. Thus, we should keep in mind that a model is only a simplification of a real world problem, and the two are not same. It is some thing like the difference between a map that gives the physical features of a country, and the country itself. We can find the height of a place above the sea level from this map, but we cannot find the characteristics of the people from it. So, we should use a model only for the purpose it is supposed to serve, remembering all the factors we have neglected while constructing it. We should apply the model only within the limits where it is applicable.

A.I.6 To What Extent We Should Try To Improve Our Model?

To improve a model we need to take into account several additional factors. When we do this we add more variables to our mathematical equations. The equations becomes complicated and the model is difficult to use. A model must be simple enough to use yet accurate; i.e the closer it is to reality the better the model is.

Try This

A problem dating back to the early 13^{th} century, posed by Leonardo Fibonacci, asks how many rabbits you would have in one year if you started with just two and let all of them reproduce. Assume that a pair of rabbits produces a pair of offspring each month and that each pair of rabbits produces their first offspring at the age of 2 months. Month by month, the number of pairs of rabbits is given by the sum of the rabbits in the two preceding months, except for the 0^{th} and the 1^{st} months. The table below shows how the rabbit population keeps increasing every month.

Month	Pairs of Rabbits
0	1
1	1
2	2
3	3
4	5
5	8
6	13
7	21
8	34
9	55
10	89
11	144
12	233
13	377
14	610
15	987
16	1597

After one year, we have 233 rabbits. After just 16 months, we have nearly 1600 pairs of rabbits.

Clearly state the problem and the different stages of mathematical modelling in this situation.

We will finish this chapter by looking at some interesting examples.

Example-4. **(Rolling of a pair of dice) :** Deekshitha and Ashish are playing with dice. Then Ashish said that, if she correctly guess the sum of numbers that show up on the dice, he would give a prize for every answer to her. What numbers would be the best guess for Deekshitha.

Solution :

Step 1 (Understanding the problem) : You need to know a few numbers which have higher chances of showing up.

Step 2 (Mathematical description) : In mathematical terms, the problem translates to finding out the probabilities of the various possible sums of numbers that the dice could show.

We can model the situation very simply by representing a roll of the dice as a random choice of one of the following thirty six pairs of numbers.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The first number in each pair represents the number showing on the first die, and the second number is the number showing on the second die.

Step 3 (Solving the mathematical problem) : Summing the numbers in each pair above, we find that possible sums are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12. We have to find the probability for each of them, assuming all 36 pairs are equally likely.

We do this in the following table.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Observe that the chance of getting a sum of a seven is $\frac{1}{6}$, which is larger than the chances of getting other numbers as sums.

Step 4 (Interpreting the solution) : Since the probability of getting the sum 7 is the highest, you should repeatedly guess the number seven.

Step 5 (Validating the model) : Toss a pair of dice a large number of times and prepare a relative frequency table. Compare the relative frequencies with the corresponding probabilities. If these are not close, then possibly the dice are biased. Then, we could obtain data to evaluate the number towards which the bias is.

Before going to the next try this exercise, we need some background information.

Not having the money you want when you need it, is a common experience for many people. Whether it is having enough money for buying essentials for daily living, or for buying comforts, we always require money. To enable the customers with limited funds to purchase goods like scooters, refrigerators, televisions, cars, etc., a scheme known as an instalment scheme (or plan) is introduced by traders.

Sometimes a trader introduces an instalment scheme as a marketing strategy to allow customers to purchase these articles. Under the instalment scheme, the customer is not required to make full payment of the article at the time of buying it. She/he is allowed to pay a part of it at the time of purchase and the rest can be paid in instalments, which could be monthly, quarterly, half-yearly, or even yearly. Of course, the buyer will have to pay more in the instalment plan, because the seller is going to charge some interest on account of the payment made at a later date (called deferred payment).

There are some frequently used terms related to this concept. You may be familiar with them. For example, the cash price of an article is the amount which a customer has to pay as full payment of the article at the time it is purchased. Cash down payment is the amount which a customer has to pay as part payment of the price of an article at the time of purchase.

Now, try to solve the problem given below by using mathematical modelling.

TRY THIS

Ravi wants to buy a bicycle. He goes to the market and finds that the bicycle of his choice costs ₹2,400. He has only ₹1,400 with him. To help, the shopkeepr offers to help him. He says that He can make a down payment of ₹1400 and pay the rest in monthly instalments of ₹550 each. Ravi can either take the shopkeepers offer or go to a bank and take a loan at 12% per annum simple interest. From these two opportunities which is the best one to Ravi. Help him.